Spring 2009

1. The midpoint is the average of the coordinates of the endpoints, so the distance from one endpoint to the midpoint is the same as the distance from the midpoint to the other endpoint.

7-3=4	find distance in x between given point and midpoint
7+4=11	add difference to midpoint x coordinate (get on other side of midpoint)
5-(-3)=8	
5+8=13	same steps with y coordinate

So the answer is **A. (11,13)**

2.

$x\Delta(x-1) = 323$	
$x(x-1)+(x-1) = 323$	definition of Δ operator
$(x+1)(x-1) = 323$	combine (x-1) terms
$x^2 - 1 = 323$	FOIL (conjugate cancels middle terms, leaving difference of 2 perfect squares)
$x^2 = 324$	add 1 to both sides
$x = \pm 18$	square root both sides
$x\Delta(x+1)$	
$x(x+1)+(x+1)$	definition
$(x+1)(x+1)$	combine (x+1) terms
$(x+1)^2$	
$(18+1)^2$	substitute 18 for x (assuming positive)
19^2	
361	

So the answer is **E. 361**

3.

2L+2W=36 perimeter

$L^2 + W^2 = 170$ diagonal (Pythagorean Theorem)

$L + W = 18$ first equation /2

$(L + W)^2 = 324$ square both sides

$L^2 + 2LW + W^2 = 324$ FOIL

$170 + 2LW = 324$ substitute using 2nd equation

$2LW = 154$ -170 both sides

$LW = 77$ /2 both sides, LW=area

So the answer is **D. 77**

4. Test each function.

$f(x) = x$

$f\big(x + f(x)\big) = f\big(f(x)\big) + f(x)$

$f(x + x) = f(x) + x$

$f(2x) = x + x$

$2x = 2x$ f(x)=x works

$f(x) = 2x$

$f\big(x + f(x)\big) = f\big(f(x)\big) + f(x)$

$f(x + 2x) = f(2x) + 2x$

$f(3x) = 4x + 2x$

$6x = 6x$ f(x)=2x works

$f(x) = \ln(x)$

$f\big(x + f(x)\big) = f\big(f(x)\big) + f(x)$

$f(x + \ln(x) = f(\ln(x)) + \ln(x))$

$\ln(x + \ln(x)) = \ln(\ln(x)) + \ln(x)$ looks like it won't work…

$\ln(x + \ln(x)) = \ln(x \ln(x))$

$x + \ln(x) = x \ln(x)$ f(x)=ln(x) does not work

So the answer is **D. A and B**

5.

$x\sqrt{14} + 7 = kx^2$

$kx^2 - \sqrt{14}x - 7 = 0$

$14 - 4(k)(-7) > 0$ 2 real solutions when determinant $(b^2 - 4ac) > 0$

$28k > -14$ -14 both sides

$k > -\frac{1}{2}$ /28 both sides, reduce

So the answer is **B.** $\boldsymbol{k > -\frac{1}{2}}$

6. 2009 factors to $7 * 7 * 41$

$x^n - x^{n-1} - x^{n-2}$ factors to $x^{n-2}(x^2 - x - 1)$

x^{n-2} has to be a factor of 2009, with $x^2 - x - 1$ being $\frac{2009}{x^{n-2}}$

The most common factor in 2009 is 7, so let's try that first.

$x = 7$

$(7^2 - 7 - 1)(7^{n-2}) = 2009$ replace 7 into $x^{n-2}(x^2 - x - 1) = 2009$

$(41)(7^{n-2}) = 2009$ arithmetic

$(7^{n-2}) = 49$ divide 41

$n - 2 = 2$ $\log_7$ both sides

$n = 4$ n is a positive integer less than x, so 7 works

$x + n = 7 + 4 = 11$

So the answer is **B. 11**

7.

What we want to know is $\frac{\frac{3}{7}W+\frac{1}{2}M}{W+M} = \frac{those\ matched\ with\ opposite\ gender}{all\ players}$

$\frac{3}{7}W = \frac{1}{2}M$

$\frac{6}{7}W = M$

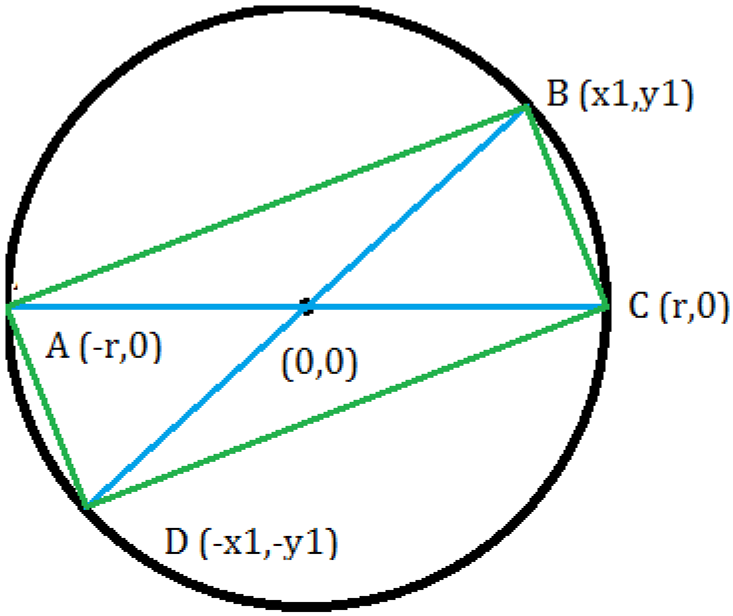
$\frac{\frac{3}{7}W+\frac{3}{7}W}{W+\frac{6}{7}W}$ substitution

$\frac{\frac{6}{7}W}{\frac{13}{7}W}$

$\frac{6}{13}$ cancel 1/7 and W

So the answer is **D. 6/13**

8.



Let's give the center coordinates (0,0) and assume one of the lines is horizontal for convenience
(any other case can be obtained by rotating)

Any time a line segment has endpoints on a circle and passes through the center, it is a diameter, so the center is a
midpoint to both lines. Since we are calling the center (0,0), the endpoints must have opposite coordinates of each other. For the horizontal line, the coordinates of the endpoints are (-r,0) and (r,0). For our other line, we can call the endpoints (x,y) and(-x,-y).

Let's look at ΔABC.

$AB = \sqrt{(x+r)^2 + (y-0)^2}$ distance formula

$AB = \sqrt{x^2 + 2xr + r^2 + y^2}$

$AB = \sqrt{x^2 + y^2 + r^2 + 2xr}$

$AB = \sqrt{2r^2 + 2xr}$ $x^2 + y^2 = r^2$ for any point on a circle

$BC = \sqrt{(x-r)^2 + (y-0)^2}$

$BC = \sqrt{x^2 - 2xr + r^2 + y^2}$

$BC = \sqrt{2r^2 - 2xr}$

$AC = 2r$

$BC^2 + AB^2 = AC^2$ test whether ABC is a right angle by Pythagorean theorem

$\sqrt{2r^2 - 2xr}^2 + \sqrt{2r^2 + 2xr}^2 = (2r)^2$

$2r^2 - 2xr + 2r^2 + 2xr = 4r^2$

$4r^2 = 4r^2$ Pythagorean Theorem holds, thus ABC is a right angle

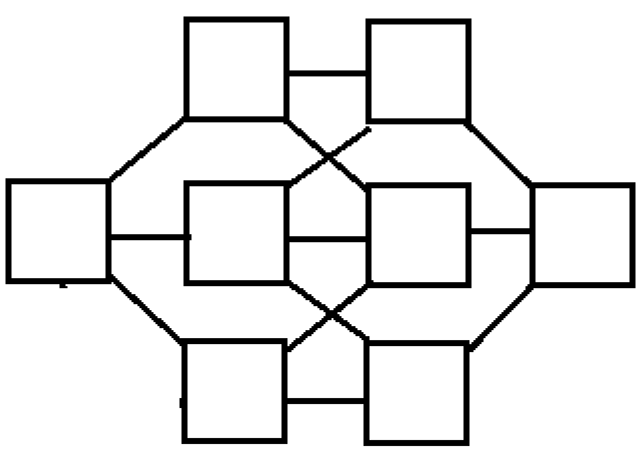
ADC, DAB, and DCB are similar to ABC, so all angles are right angles

Also, opposite sides have the same length, but adjacent sides do not, unless x=0, so ABCD is not necessarily a square.

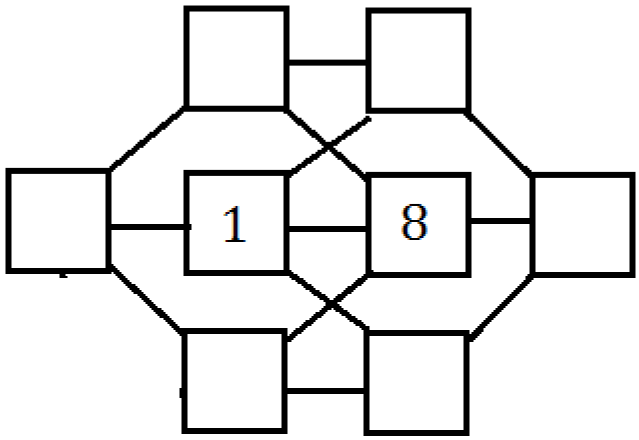
So ABCD must be a rectangle

So the answer is **C. Rectangle**

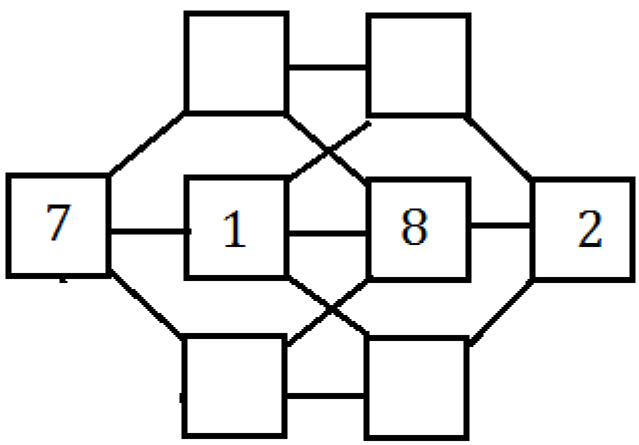
9.



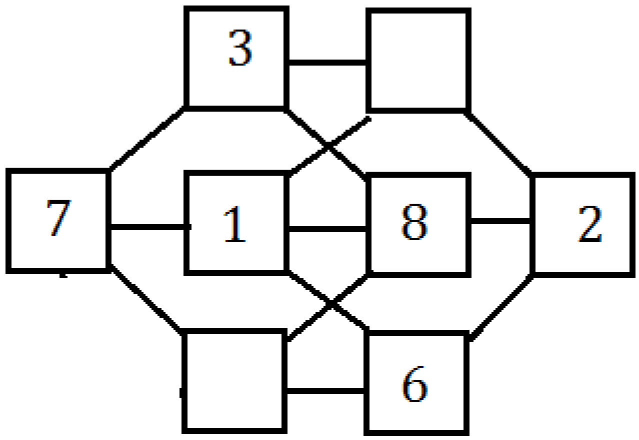
The middle 2 squares each touch all but one of the other squares, so only 1 and 8 can go in them.

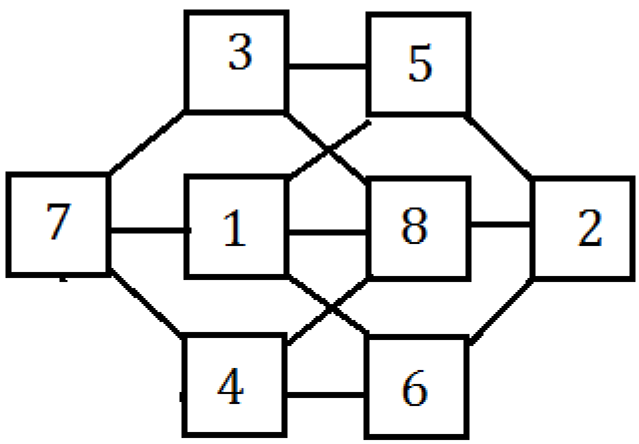


2 and 7 must not touch 1 and 8, respectively, so their places are determined



3 cannot touch 2 and 6 cannot touch 7, but they need to be split so 4 and 5 can be split





Adding up the indicated spots (or any 2 opposite spots) gets us 9

So the answer is **C. 9**

10.

$V = \frac{1}{3}\pi r^2 h$

The radius is proportional to the height (by similar triangles), so the new volume will be proportional to the radius cubed

Therefore, to cut the volume in half, the radius should be divided by the cube root of 2

The new radius will therefore be $\frac{4}{\sqrt[3]{2}}$

$\frac{4}{\sqrt[3]{2}} * \frac{\sqrt[3]{4}}{\sqrt[3]{4}}$ rationalize the denominator

$\frac{4\sqrt[3]{4}}{2}$

$2\sqrt[3]{4}$

So the answer is **A.** $\mathbf{2\sqrt[3]{4}}$

11. A = # of steps above, B = # of steps below

$A = 2B$

$A - 5 = B + 5$ after climbing 5 steps

$A = B + 10$ add 5 both sides

$2B = B + 10$ substitution for A

$B = 10$ subtract B both sides

$A = 2(10)$ substitution

$A = 20$

$4(A - 5 - x) = B + 5 + x$ x # of steps climbed for # below to be 4 * # above

$4(15 - x) = 15 + x$ substitute A and B

$60 - 4x = 15 + x$ distribute

$45 = 5x$ combine like terms

$x = 9$ divide by 5

So the answer is **E. 9**

12.

$\sin^3\theta - \cos^3\theta$

$(\sin\theta - \cos\theta)(\sin^2\theta + \sin\theta\cos\theta + \cos^2\theta)$ difference of 2 cubes expansion

$(\sin\theta - \cos\theta)(\sin^2\theta + \cos^2\theta + \frac{1}{2}\sin 2\theta)$ $\sin 2\theta = 2\sin\theta\cos\theta$

$(.2)(1+.48)$ $\sin^2\theta + \cos^2\theta = 1$, others are given

$(.2)(1.48)$

$.296$

So the answer is **D. .296**

13.

The vertical asymptotes occur when the denominator = 0

$10\sqrt{100x^2-1} = 0$

$100x^2 - 1 = 0$

$100x^2 = 1$

$10x = \pm 1$

$x = \frac{1}{10}, -\frac{1}{10}$

That's 2 asymptotes so far.

Now let's consider horizontal asymptotes.

$\sqrt{100x^2-1}$ is always slightly less than |10x|

So as x gets larger, $\sqrt{100x^2-1}$ gets closer to 10x, and $\frac{x}{10\sqrt{100x^2-1}}$ gets closer to .01

And as x gets larger in the negative direction, $\sqrt{100x^2-1}$ gets closer to -10x and $\frac{x}{10\sqrt{100x^2-1}}$ gets closer to -.01.

So we have asymptotes at y=.01, -.01

This gives us a total of 4 asymptotes

So the answer is **E. 4**

14.

$x^2 = x^2$

$(x+1)^2 = x^2 + 2x + 1$

$(x+2)^2 = x^2 + 4x + 4$

$(x+3)^2 = x^2 + 6x + 9$

$x^2 + 4x + 6$ is not a perfect square, so it is never y^2 if y is an integer

So the answer is **A. 0**

15.

$685 = 137 * 5$

$5 = 2^2 + 1^2$

$137 = 11^2 + 4^2$ can use $y = \sqrt{137 - x^2}$ in calculator to find

$685 = (2^2 + 1^2)(11^2 + 4^2)$ substitution

$= 22^2 + 8^2 + 11^2 + 4^2$ FOIL

$8 * 11 = 4 * 22$ $rs = tu$

(negative doesn't matter because of abs values and squares)

$8 + 11 + 4 + 22 = 45$

So the answer is 45, but AMATYC gave the answer to all students free.

16.

The first light can be any color.

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(1)=   G    R    Y          =3
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G leads to Y or R, R leads to Y, and Y leads to the original set (1)

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(2)=  R Y   Y   (1)         =6
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R leads to Y, Y leads to the original set, and (1) leads to (2)

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(3)= Y  (1) (1)  (2)        =13
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Y leads to (1), and the rest increase by 1 step

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(4)= (1) (2) (2) (3)        =28
(5)= (2) (3) (3) (4)        =60
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So the answer is **D. 60**

17.

$2x^2y + 2xy = 0, y \neq 0$

$x^2 + x = 0$ divide by 2y

$x(x+1) = 0$ factor

$x = 0, x = -1$

$6(-1)^2y + y^3 + 10(-1)y = 0$ substitute -1

$6y + y^3 - 10y = 0$

$y^3 - 4y = 0$ combine like terms

$y(y+2)(y-2) = 0$ factor

$y = -2,2$ two solutions ($y \neq 0$)

$6(0)^2y + y^3 + 10(0)y = 0$

$y^3 = 0$ since $y \neq 0$, no solutions

total of 2 solutions

So the answer is **B. 2**

18.

$x + y = x^3 + y^3$

$x + y = (x+y)(x^2 - xy + y^2)$ factor (sum of 2 cubes)

$0 = (x+y)(x^2 - xy + y^2) - (x+y)$

$0 = (x+y)(x^2 - xy + y^2 - 1)$

$x + y = 0$ line

OR

$x^2 - xy + y^2 - 1 = 0$

$x^2 - xy + y^2 = 1$ rotated ellipse

So the answer is **A. line and an ellipse**

19.

Combinations of 4 numbers of 1, 5, and 9 that add up to 16 are 1 “1” and 3 “5”’s, or 2 “1”’s 1 “5”, and 1 “9”.

Combinations of a 1 and 3 5’s are 1555, 5155, 5515, and 5551, none of which are multiples of 37

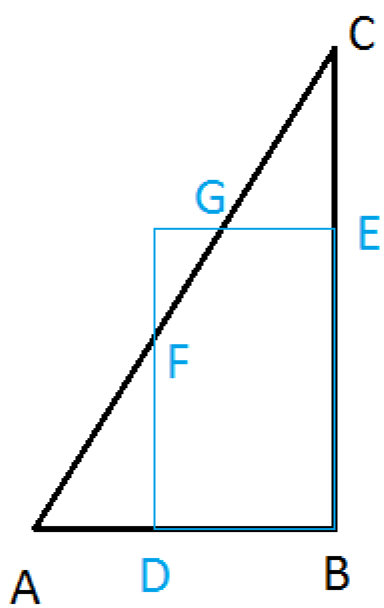
Combinations of 2 “1”’s, 1 “5”, and 1 “9” are 1159, 1195, 1519, 1591, 1915, 1951, 5119, 5191, 5911, 9115, 9151, and 9511.

1591 is the only one of those that is a multiple of 37

9+1=10

So the answer is **C. 10**

20.



We can get the area shared by the triangle and rectangle by finding the area of the triangle, then subtracting the areas of the smaller triangles outside the rectangle.

The area of $\Delta ABC = \frac{1}{2}bh = \frac{1}{2} * 5 * 8 = 20$

ΔADF is similar, with $\frac{2}{5}$ the base of ΔABC, and ΔGEC is similar with $\frac{3}{8}$ the height, so

$$\Delta ADF = \frac{1}{2} * (\frac{2}{5} * 5) * (\frac{2}{5} * 8) = \frac{16}{5}$$

$$\Delta GEC = \frac{1}{2} * \left(\frac{3}{8} * 5\right) * \left(\frac{3}{8} * 8\right) = \frac{45}{16}$$

$$\Delta ABC - \Delta ADF - \Delta GEC = 20 - \frac{16}{5} - \frac{45}{16} = \frac{1600}{80} - \frac{256}{80} - \frac{225}{80} = \frac{1119}{80}$$

So the answer is **D. 1119/80**